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On the Generalisation of the Theory of the Spin Maximum 1 Particle to the Case of a Charged Particle Moving in an Electromagnetic Field

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1. *Preliminary Considerations About Compatibility*

When one wants to generalise most of the equations of evolution in wave mechanics in order to embrace the case of electrically charged particles moving within an electromagnetic field, one usually employs a procedure whose origins may be found in analytical dynamics and which consists of replacing the momentum and energy operators, namely

$$\mathbf{p}_{\text{op}} = i\hbar\nabla, \qquad E_{\text{op}} = -i\hbar\partial_t \tag{1.1}$$

by

$$\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)_{\text{op}} = i\hbar\nabla - \frac{q}{c}\mathbf{A}, \qquad (E - qV)_{\text{op}} = -i\hbar\partial_t - qV \tag{1.2}$$

respectively. (In the preceding, q denotes the electric charge of the particle and $\mathbf{A}$, V are the electromagnetic potentials of the exterior field.) If, for instance, we make use of this method in the particular case of the Dirac equation,

$$\left[\frac{E}{c} + \mathbf{p}\,.\,\boldsymbol{\alpha} - m_0\,c\alpha_4\right]_{\text{op}} \psi = 0$$

we then get the well known generalised equations

$$\left[\frac{1}{c}(E - qV) + \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right).\,\boldsymbol{\alpha} - m_0\,c\alpha_4\right]_{\text{op}} \psi = 0$$

The problem then arises of seeking for a similar generalisation for the relativistic equations of the particle with spin maximum 1 (de Broglie, 1940) which can be written in the form

$$\begin{gathered}\left[\frac{1}{c}\partial_t - \partial_x a_1 - \partial_y a_2 - \partial_z a_3 - \frac{im_0 c}{\hbar} a_4\right]\psi = 0 \\ \left[\frac{1}{c}\partial_t - \partial_x b_1 - \partial_y b_2 - \partial_z b_3 - \frac{im_0 c}{\hbar} b_4\right]\psi = 0,\end{gathered} \tag{1.3}$$

where m_0 is the proper mass of the particle and ψ is a column matrix with sixteen elements $\psi_{11}\psi_{12}\psi_{13}\psi_{14}\psi_{21}\psi_{22}\ldots\psi_{42}\psi_{43}\psi_{44}$. As for the a_μ, b_μ, they are 16×16 matrices built with the 4×4 identity matrix I and the four Dirac matrices α_μ in the following way

$$a_\mu = \alpha_\mu \times I \qquad b_\mu = I \times \alpha_\mu, \qquad (\mu = 1, 2, 3, 4) \tag{1.4}$$

where we denote by $A \times B$ the external product of the two matrices A and B, obtained by replacing each element a_{ik} of A by the matrix $a_{ik}B$.

If we introduce into (1.3) the energy and momentum operators defined in (1.1), the equations become

$$\begin{gathered}\left[\frac{E}{c} + \mathbf{p}.\mathbf{a} - m_0 c a_4\right]_{\text{op}} \psi = 0 \\ \left[\frac{E}{c} + \mathbf{p}.\mathbf{b} - m_0 c b_4\right]_{\text{op}} \psi = 0.\end{gathered}$$

Now it can be seen that if the former substitution of (1.1) by (1.2) is performed upon the preceding equations the resulting system,

$$\left[\frac{1}{c}(E - qV) + \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right).\mathbf{a} - m_0 c a_4\right]_{\text{op}} \psi = 0 \tag{1.5a}$$

$$\left[\frac{1}{c}(E - qV) + \left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right).\mathbf{b} - m_0 c b_4\right]_{\text{op}} \psi = 0 \tag{1.5b}$$

is incompatible. To make sure, sum and subtract the equations (1.5), which gives

$$\left[\frac{-i\hbar\partial_t - qV}{c} + \left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a} + \mathbf{b}}{2} - m_0 c\frac{a_4 + b_4}{2}\right]\psi = 0 \tag{1.6a}$$

$$\left[\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a} - \mathbf{b}}{2} - m_0 c\frac{a_4 - b_4}{2}\right]\psi = 0 \tag{1.6b}$$

that is, sixteen equations (1.6a) having first derivatives of time and sixteen equations (1.6b) where those derivatives do not occur. We shall now see that on account of equations (1.6a), the equations (1.6b) do not remain valid over time, which amounts to saying that

$$\partial_t\left\{\left[\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a} - \mathbf{b}}{2} - m_0 c\frac{a_4 - b_4}{2}\right]\psi\right\} \neq 0.$$

The preceding derivative is equal to

$$-\frac{q}{c}(\partial_t \mathbf{A}).\frac{\mathbf{a}-\mathbf{b}}{2}\psi + \left[\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a}-\mathbf{b}}{2} - m_0 c\frac{a_4 - b_4}{2}\right]\partial_t \psi$$

and if we take into account the value of $\partial_t \psi$ obtained from (1.6a), we are led to the equivalent expression

$$\begin{aligned}
&-\frac{q}{c}(\partial_t \mathbf{A}).\frac{\mathbf{a}-\mathbf{b}}{2}\psi - \frac{q}{i\hbar}\left[\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a}-\mathbf{b}}{2} - m_0 c\frac{a_4 - b_4}{2}\right]V\psi \\
&+\frac{c}{i\hbar}\left[\sum_{k=1}^{3}\left(i\hbar\partial_k - \frac{q}{c}A_k\right)^2 \frac{a_k - b_k}{2}.\frac{a_k + b_k}{2}\right. \\
&+\sum_{l\neq k}^{3}\left(i\hbar\partial_l - \frac{q}{c}A_l\right)\left(i\hbar\partial_k - \frac{q}{c}A_k\right)\frac{a_l - b_l}{2}\frac{a_k + b_k}{2} \\
&-m_0 c\sum_{k=1}^{3}\left(i\hbar\partial_k - \frac{q}{c}A_k\right)\left(\frac{a_k - b_k}{2}\frac{a_4 + b_4}{2} + \frac{a_4 - b_4}{2}\frac{a_k + b_k}{2}\right) \\
&\left.+ m_0^2 c^2 \frac{a_4 - b_4}{2}\frac{a_4 + b_4}{2}\right]\psi \qquad (1.7)
\end{aligned}$$

Now by means of definition (1.4) and the multiplication properties of the α_μ $(\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu} I)$, the following relation can be verified

$$(a_\mu - b_\mu)(a_\nu + b_\nu) + (a_\nu - b_\nu)(a_\mu + b_\mu) = 0 \qquad (\mu, \nu = 1, 2, 3, 4)$$

and furthermore one easily recognises that

$$\left(i\hbar\partial_l - \frac{q}{c}A_l\right)\left(i\hbar\partial_k - \frac{q}{c}A_k\right) - \left(i\hbar\partial_k - \frac{q}{c}A_k\right)\left(i\hbar\partial_l - \frac{q}{c}A_l\right) = i\hbar\frac{q}{c}(\operatorname{curl}\mathbf{A})_m$$

(l, k, m being a circular permutation of 1, 2, 3), so that (1.7) takes the form

$$\begin{aligned}
&-\frac{q}{c}(\partial_t \mathbf{A}).\frac{\mathbf{a}-\mathbf{b}}{2}\psi - \frac{q}{i\hbar}\left[\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right).\frac{\mathbf{a}-\mathbf{b}}{2} - m_0 c\frac{a_4 - b_4}{2}\right]V\psi \\
&\qquad -q\left[\sum_{k,l,m}(a_l - b_l)(a_k + b_k)(\operatorname{curl}\mathbf{A})_m\right]\psi
\end{aligned}$$

To ensure this expression be null, it is then required that $\partial_t \mathbf{A} = V = \operatorname{rot}\mathbf{A} = 0$, which is equivalent to saying that the exterior electromagnetic field must be null, and we thus regress to the non-generalised equations at the beginning. One thus comes to the conclusion that equations (1.3), describing the spin maximum 1 particle, cannot undergo the usual generalisation (1.1), (1.2) without bringing about a system which is incompatible.

2. *The Lagrangian and the Generalised Equations*

Let us point out at once that the incompatibility of equations (1.5) is inherent in the non-commutability of the four operators $(E-qV)_{\text{op}}$ and $[\mathbf{p}-(q/c)\mathbf{A}]_{\text{op}}$ as well as to the very structure of (1.3) which contains thirty-two equations of the sixteen functions ψ_{ik}. The system nevertheless remains compatible in course of time as in de Broglie, 1954, but if we carry out the generalisation (1.1), (1.2) upon the equations (1.3) we may expect their compatibility to fail and, according to our discussion in Section 1, this is what actually happens. As has been shown (see Pereira, 1971) this fact means that among the thirty-two equations of evolution and condition describing the behaviour of the spin maximum 1 particle, there will be a certain number of equations which are no longer valid in the generalised case of an electrically charged particle.

In order to overcome this difficulty we shall consider the following system of sixteen first-order differential equations

$$\left[\frac{1}{c}\partial_t\frac{a_4+b_4}{2}-\sum_k^3\partial_k\frac{a_4b_k+a_kb_4}{2}-\frac{im_0c}{\hbar}a_4b_4\right]\psi=0.$$

It has been proved in de Broglie (1957) that this system is equivalent to (1.3) and that equations (2.1) are precisely the sixteen Lagrange equations arising from the Lagrangian

$$\mathscr{L}_0=\frac{\hbar c}{2i}\psi^*\left[\frac{1}{c}\partial_t\frac{a_4+b_4}{2}-\nabla.\frac{a_4\mathbf{b}+\mathbf{a}b_4}{2}-\frac{im_0c}{\hbar}a_4b_4\right]\psi+\text{conj}.$$

Now it is well known that many expressions in the theory of the particle with spin maximum 1 can be formally obtained from the corresponding expressions of the Dirac theory just by carrying on the substitution of matrices

$$I\to\frac{a_4+b_4}{2}$$

$$\alpha_\mu\to\frac{a_\mu b_4+a_4b_\mu}{2}\qquad(\mu=1,2,3,4).$$

Thus, it seems natural to believe that if we perform this substitution upon the generalised Lagrangian of Dirac, namely

$$\mathscr{L}^D=\frac{\hbar c}{2i}\psi^*\left[\left(\frac{1}{c}\partial_t-\frac{iq}{\hbar c}V\right)-\left(\nabla+\frac{iq}{\hbar c}\mathbf{A}\right).\boldsymbol{\alpha}-\frac{im_0c}{\hbar}\alpha_4\right]\psi+\text{conj}.$$

the resulting expression,

$$\mathscr{L}=\frac{\hbar c}{2i}\psi^*\left[\left(\frac{1}{c}\partial_t-\frac{iq}{\hbar c}V\right)\frac{a_4+b_4}{2}-\left(\nabla+\frac{iq}{\hbar c}\mathbf{A}\right).\frac{\mathbf{a}b_4+a_4\mathbf{b}}{2}-\frac{im_0c}{\hbar}a_4b_4\right]\psi+\text{conj}. \tag{2.2}$$

will be the Lagrangian of the particle with spin maximum 1 and electric charge q moving in an electromagnetic field defined by the potentials A, V. From (2.2) we may calculate the derivatives

$$\frac{\partial \mathscr{L}}{\partial(\partial_k \psi_{\mu\nu})} = -\frac{\hbar c}{2i}\left(\frac{a_k b_4 + b_k a_4}{2}\right)^* \psi^* \qquad \frac{\partial \mathscr{L}}{\partial(\partial_t \psi_{\mu\nu})} = \frac{\hbar}{2i}\left(\frac{a_4 + b_4}{2}\right)^* \psi^*$$

$$\frac{\partial \mathscr{L}}{\partial \psi_{\mu\nu}} = -\frac{\hbar c}{2i}\left[\left(\frac{1}{c}\partial_t + \frac{2iq}{\hbar c}V\right)\left(\frac{a_4 + b_4}{2}\right)^* - \left(\nabla - \frac{2iq}{\hbar c}\mathbf{A}\right).\left(\frac{\mathbf{a}b_4 + a_4\mathbf{b}}{2}\right)^* + \frac{2im_0 c}{\hbar}(a_4 b_4)^*\right]\psi^*$$

and by introducing the preceding expressions in the Lagrange equations

$$\partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \psi_{\alpha\beta})} + \sum_{k=1}^{3} \partial_k \frac{\partial \mathscr{L}}{\partial(\partial_k \psi_{\alpha\beta})} = \frac{\partial \mathscr{L}}{\partial \psi_{\alpha\beta}} \qquad (\alpha, \beta = 1,2,3,4) \tag{2.3}$$

one is led to the relations

$$\left[\left(\frac{1}{c}\partial_t - \frac{iq}{\hbar c}V\right)\frac{a_4 + b_4}{2} - \sum_{k=1}^{3}\left(\partial_k + \frac{iq}{\hbar c}A_k\right)\frac{a_k b_4 + a_4 b_k}{2} - \frac{im_0 c}{\hbar}a_4 b_4\right]\psi = 0. \tag{2.4}$$

These are the spinor equations which we shall adopt from now on as describing the spin maximum 1 particle of electric charge q moving in an electromagnetic field given by the potentials $\mathbf{A}$ and V. From its very structure, the preceding system (with sixteen functions $\psi_{\alpha\beta}$ obeying to sixteen first-order differential equations) does not give rise to any sort of difficulty, as far as mathematical compatibility is concerned. Moreover, if we put $\mathbf{A} = V = 0$ (or $q = 0$) we obtain the system (2.1) which yields (1.3) as we have referred to in the foregoing.

3. *The 4-Vector Current Flow Density and the Energy-Momentum Density Tensor*

We must now examine whether the existence of q, $\mathbf{A}$ and V brings about some alterations in the expressions of the physical quantities attached to the particle with regard to the same physical quantities when $\mathbf{A} = V = 0$.

We may say at once that the definition of the 4-vector probability flow-density remains unchanged. As a matter of fact, one may easily verify that an equation of continuity

$$\partial_t \rho + \operatorname{div} \mathbf{f} = 0 \tag{3.1}$$

is still implied by equations (2.4) with the usual definitions

$$\rho = \psi^* \frac{a_4 + b_4}{2} \psi$$

$$\mathbf{f} = -c\psi^* \frac{\mathbf{a}b_4 + a_4\mathbf{b}}{2} \psi,$$

$i\rho$ and $\mathbf{f}/c$ being the components of a 4-vector which we denote by j_μ.† Since the particle is now supposed to be endowed with an electric charge q, we may define the 4-vector current-flow density $s_\mu = qj_\mu = [iq\rho; q\mathbf{f}/c]$, so that the equation of continuity (3.1) becomes

$$\partial_\mu s_\mu = 0 \tag{3.2}$$

which is the expression of the conservation of the total charge.

Now the 4-vector s_μ can be written in a form which will be useful later. In fact, the Lagrangian (2.2) yields

$$\frac{\partial \mathscr{L}}{\partial A_k} = -q\psi^* \frac{a_4 b_k + a_k b_4}{2} \psi \qquad (k = 1, 2, 3)$$

$$\frac{\partial \mathscr{L}}{\partial V} = -q\psi^* \frac{a_4 + b_4}{2} \psi$$

and by introducing the 4-vector $\mathscr{A}_\mu = [iV; \mathbf{A}]$, one then obtains

$$s_\mu = qj_\mu = \frac{\partial \mathscr{L}}{\partial \mathscr{A}}. \tag{3.3}$$

Let us now examine the energy-momentum density tensor $T_{\mu\nu}$ of the particle of spin maximum 1 and charge q. If the particle is moving in an electromagnetic field $\mathbf{A}$, V, this tensor must verify $\partial_\mu T_{\mu\nu} = r_\nu$, where r_ν is a 4-vector whose space components are the three components of the Lorentz force $q\rho[\mathbf{E} - (1/c)\mathbf{H} \wedge \mathbf{v}]$, and whose time component is the work $(i/c)q\rho\mathbf{E}.\mathbf{v}$. With our relativistic notations,‡ this may be expressed in the form

$$\partial_\mu T_{\mu\nu} = \mathscr{H}_{\nu\rho} s_\rho = (\partial_\nu \mathscr{A}_\rho - \partial_\rho \mathscr{A}_\nu) s_\rho. \tag{3.4}$$

Starting from this relation we shall now seek for the explicit form of $T_{\mu\nu}$. If we calculate the derivative of the Lagrangian in order to the x_β coordinate we obtain, by means of the Lagrange equations (2.3),

$$\begin{aligned}\partial_\beta \mathscr{L} = \partial_\sigma(\delta_{\sigma\beta} \mathscr{L}) &= \frac{\partial \mathscr{L}}{\partial(\partial_\sigma \psi_{\mu\nu})} \partial^2_{\sigma\beta} \psi_{\mu\nu} + \frac{\partial \mathscr{L}}{\partial \psi_{\mu\nu}} \partial_\beta \psi_{\mu\nu} + \text{conj.} + \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\sigma} \partial_\beta \mathscr{A}_\sigma \\ &= \frac{\partial \mathscr{L}}{\partial(\partial_\sigma \psi_{\mu\nu})} \partial^2_{\sigma\beta} \psi_{\mu\nu} + \partial_\sigma \left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma \psi_{\mu\nu})} \right) \partial_\beta \psi_{\mu\nu} + \text{conj.} \\ &\quad + \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\sigma} \partial_\beta \mathscr{A}_\sigma\end{aligned}$$

and hence

$$\frac{\partial \mathscr{L}}{\partial \mathscr{A}_\sigma} \partial_\beta \mathscr{A}_\sigma = \partial_\sigma \left[\delta_{\alpha\beta} \mathscr{L} - \frac{\partial \mathscr{L}}{\partial(\partial_\sigma \psi_{\mu\nu})} \partial_\beta \psi_{\mu\nu} + \text{conj.} \right]. \tag{3.5}$$

† In the sequel we shall always make use of the coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$.

‡ We define the electromagnetic field tensor as follows, $\mathscr{H}_{14} = -iE_x$, $\mathscr{H}_{24} = -iE_y$, $\mathscr{H}_{34} = -iE_z$; $\mathscr{H}_{12} = H_z$, $\mathscr{H}_{23} = H_x$, $\mathscr{H}_{31} = H_y$; ($\mathscr{H}_{\mu\nu} = -\mathscr{H}_{\nu\mu}$). The definition relations of the fields, $\mathbf{E} = -(1/c)\partial_t \mathbf{A} - \operatorname{grad} V$ and $\mathbf{H} = \operatorname{rot} \mathbf{A}$, will therefore be written $\mathscr{H}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu$.

Let us now return to equation (3.4) and write, according to (3.2),

$$\partial_\alpha T_{\alpha\beta} = s_\sigma \partial_\beta \mathscr{A}_\sigma - \mathscr{A}_\beta \partial_\sigma s_\sigma - s_\sigma \partial_\sigma \mathscr{A}_\beta = s_\sigma \partial_\beta \mathscr{A}_\sigma - \partial_\sigma(s_\sigma \mathscr{A}_\beta)$$

so that one gets from (3.3) and (3.5),

$$\partial_\alpha T_{\alpha\beta} = \partial_\sigma \left[\delta_{\sigma\beta} \mathscr{L} - \left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma \psi_{\mu\nu})} \partial_\beta \psi_{\mu\nu} + \text{conj.} \right) - \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\sigma} \mathscr{A}_\beta \right].$$

This leads us to define the energy-momentum density tensor of the particle with spin maximum 1 and charge q in an electromagnetic field of 4-potential $\mathscr{A}_\mu$ in the following way,

$$T_{\alpha\beta} = -\frac{\partial \mathscr{L}}{\partial(\partial_\alpha \psi_{\mu\nu})} \partial_\beta \psi_{\mu\nu} - \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \psi^*_{\mu\nu})} \partial_\beta \psi^*_{\mu\nu} + \delta_{\alpha\beta} \mathscr{L} - \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\alpha} \mathscr{A}_\beta. \quad (3.6)$$

If we put $\mathscr{A}_\mu = 0$ (or $q = 0$), the preceding asymmetric tensor reduces to the usual form of $T_{\mu\nu}$ given by de Broglie (1940), from which it differs by the adjunction of the term $-s_\alpha \mathscr{A}_\beta$.

4. *On the Tensorial Form of the Generalised Equations*

We shall now look for the tensorial form of the spinor equations (2.4) which we introduced in Section 2.

To begin with, one can easily verify that the product and commutation properties of the a_μ and b_μ bring about the relations

$$a_4 b_4 (a_4 + b_4) = a_4 + b_4$$

$$a_4 b_4 (a_k b_4 + a_4 b_k) = a_4 a_k + b_4 b_k \qquad (k = 1, 2, 3)$$

$$a_4 b_4 a_4 b_4 = I$$

Therefore, by multiplying (2.4) at the left by $-ia_4 b_4$ one obtains

$$\left[\left(\frac{1}{ic} \partial_t + \frac{iq}{\hbar c} iV \right) \frac{a_4 + b_4}{2} + i \sum_k \left(\partial_k + \frac{iq}{\hbar c} A_k \right) \frac{a_4 a_k + b_4 b_k}{2} - \frac{m_0 c}{\hbar} \right] \psi = 0 \quad (4.1)$$

and if we define the 4-vector $P_\mu \equiv i\hbar \partial_\mu - (q/c)\mathscr{A}_\mu$, where $\mathscr{A}_\mu$ is the 4-potential and ∂_μ the 4-gradient, equation (4.1) will then take the form

$$[a_4 P_4 + \sum i a_4 a_k P_k - 2im_0 c] \psi + [b_4 P_4 + \sum i b_4 b_k P_k] \psi = 0 \qquad (4.2)$$

Let us now introduce the 4×4 matrix Ψ built with the sixteen elements $\psi_{\mu\nu}$ of the column matrix ψ:

$$\Psi \equiv \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{bmatrix}$$

By making use of (1.4) one then recognises that (4.2) can be written as follows

$$\left[\sum_\mu^4 \gamma_\mu P_\mu - 2im_0 c \right] \Psi + \left[\sum_\mu^4 \gamma_\mu P_\mu \Psi^T \right]^T = 0 \qquad (4.3)$$

where the γ_μ $(\mu = 1,2,3,4)$ are the von Neumann matrices

$$\gamma_4 = \alpha_4, \qquad \gamma_k = i\alpha_4 \alpha_k \qquad (k = 1,2,3)$$

Now let Γ be a matrix such that $\gamma_\mu{}^T \Gamma = -\Gamma\gamma_\mu$ $(\mu = 1,2,3,4)$. This matrix will obviously depend upon our choice of the γ_μ's (or, more precisely, of the α_μ's) and it can be seen that, if we fix the expression of the α_μ's as

$$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \qquad \alpha_2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \qquad \alpha_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{4.4}$$

the matrix

$$\Gamma = i\gamma_2 \gamma_4 \tag{4.5}$$

will fulfil the preceding requirements. Therefore, by multiplying (4.3) at left by Γ, we obtain the equivalent system

$$\left[\sum_\mu^4 \gamma_\mu P_\mu - 2im_0 c\right] \Psi\Gamma - \sum_\mu^4 P_\mu \Psi\Gamma\gamma_\mu = 0 \tag{4.6}$$

Let us now consider the complete system spanned by the four γ_μ, which consists of the sixteen Hermitian matrices

$$\begin{aligned} \gamma_0 &= I \\ \gamma_k &= i\alpha_4 \alpha_k \qquad (k = 1,2,3) \\ \gamma_4 &= \alpha_4 \\ \gamma_{\mu\nu} &= i\gamma_\mu \gamma_\nu \qquad (\mu < \nu;\, \mu, \nu = 1,2,3,4) \\ \gamma_{\mu\nu\rho} &= i\gamma_\mu \gamma_\nu \gamma_\rho \qquad (\mu < \nu < \rho;\, \mu, \nu, \rho = 1,2,3,4) \\ \gamma_{1234} &= \gamma_1 \gamma_2 \gamma_3 \gamma_4 \end{aligned} \tag{4.7}$$

Since any 4×4 matrix can be written as a linear combination of the preceding matrices, we shall consider the 4×4 matrix $\Psi\Gamma$ and write, with coefficients ϕ_a,

$$\Psi\Gamma = \sum_a^{16} \phi_a \gamma_a = \phi_0 \gamma_0 + \sum_\mu^4 \phi_\mu \gamma_\mu + \sum_{\mu,\nu=1,2,3,4}^{\mu<\nu} \phi_{\mu\nu} \gamma_{\mu\nu} + \sum_{\mu,\nu,\rho=1,2,3,4}^{\mu<\nu<\rho} \phi_{\mu\nu\rho} \gamma_{\mu\nu\rho} + \phi_{1234} \gamma_{1234} \dagger \tag{4.8}$$

† As can be seen in de Broglie (1954), the six functions $\phi_{\mu\nu}(\mu < \nu; \mu, \nu = 1,2,3,4)$ are the independent components of a skew-symmetric tensor of rank 2, while the $\phi_{\mu\nu\rho}$ $(\mu < \nu < \rho;$ $\mu, \nu, \rho = 1,2,3,4)$ are the four independent components of a completely skew-symmetric tensor of rank 3. Besides, ϕ_0 is an invariant and ϕ_{1234} a pseudo-invariant.

Let us now introduce (4.8) to equations (4.6) and, after a few simple calculations, equate to zero the coefficients of each matrix γ_a (for they are linearly independent). We are thus led to the following relations

$$\partial_\mu \phi_{\mu\nu} + \frac{iq}{\hbar c} \mathscr{A}_\mu \phi_{\mu\nu} = -\frac{im_0 c}{\hbar} \phi_\nu \tag{4.9a}$$

$$\partial_\mu \phi_\nu - \partial_\nu \phi_\mu + \frac{iq}{\hbar c} (\mathscr{A}_\mu \phi_\nu - \mathscr{A}_\nu \phi_\mu) = \frac{im_0 c}{\hbar} \phi_{\mu\nu} \tag{4.9b}$$

$$\phi_0 = 0 \tag{4.10a}$$

$$\partial_\mu \phi_{\mu\nu\rho\sigma} + \frac{iq}{\hbar c} \mathscr{A}_\mu \phi_{\mu\nu\rho\sigma} = \frac{im_0 c}{\hbar} \phi_{\nu\rho\sigma} \tag{4.10b}$$

$$\begin{aligned} &\partial_\mu \phi_{\nu\rho\sigma} - \partial_\nu \phi_{\rho\sigma\mu} + \partial_\rho \phi_{\sigma\mu\nu} - \partial_\sigma \phi_{\mu\nu\rho} \\ &+ \frac{iq}{\hbar c} (\mathscr{A}_\mu \phi_{\nu\rho\sigma} - \mathscr{A}_\nu \phi_{\rho\sigma\mu} + \mathscr{A}_\rho \phi_{\sigma\mu\nu} - \mathscr{A}_\sigma \phi_{\mu\nu\rho}) = -\frac{im_0 c}{\hbar} \phi_{\mu\nu\rho\sigma} \end{aligned} \tag{4.10c}$$

which are the tensor equations of the particle of spin maximum 1 and charge q moving in a field $\mathscr{A}_\mu$. It turns out that the description of the spin 0 particle and that of the spin 1 particle are completely independent. The description of the spin 0 particle is given by the last three equations whereas the first two equations concern the spin 1 particle, whose physical state is thus translated by the 4-vector ϕ_μ and the skew-symmetric tensor of rank 2, $\phi_{\mu\nu}$.

5. *The Tensorial Lagrangian Formalism*

We shall next present the tensorial form of the Lagrangian $\mathscr{L}$, that is, the expression of (2.2) in which the spinors $\psi_{\mu\nu}$ are replaced by the linear combinations of the tensor variables ϕ_a, given by equations (4.8). To begin with, we shall make use of the choice (4.4) adopted above for the α_μ's and calculate the explicit form of the matrices γ_a and Γ defined in (4.5) and (4.7). By introducing in (4.8) the matrices thus obtained, one is then led to the expressions of the $\psi_{\mu\nu}$,

$$\begin{vmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{vmatrix} = \begin{vmatrix} -\phi_1 + i\phi_2 + i\phi_{14} + \phi_{24} & \phi_3 - i\phi_{34} - \phi_{123} - i\phi_{1234} \\ \phi_3 - i\phi_{34} + \phi_{123} + i\phi_{1234} & \phi_1 + i\phi_2 - i\phi_{14} + \phi_{24} \\ -\phi_{13} + i\phi_{23} + \phi_{134} - i\phi_{234} & i\phi_0 - i\phi_4 - i\phi_{12} + i\phi_{124} \\ -i\phi_0 + i\phi_4 - i\phi_{12} + i\phi_{124} & -\phi_{13} - i\phi_{23} + \phi_{134} + i\phi_{234} \end{vmatrix}$$

$$\times \begin{vmatrix} -\phi_{13} + i\phi_{23} - \phi_{134} + i\phi_{234} & i\phi_0 + i\phi_4 - i\phi_{12} - i\phi_{124} \\ -i\phi_0 - i\phi_4 - i\phi_{12} - i\phi_{124} & -\phi_{13} - i\phi_{23} - \phi_{134} - i\phi_{234} \\ \phi_1 - i\phi_2 + i\phi_{14} + \phi_{24} & -\phi_3 - i\phi_{34} + \phi_{123} - i\phi_{1234} \\ -\phi_3 - i\phi_{34} - \phi_{123} + i\phi_{1234} & -\phi_1 - i\phi_2 - i\phi_{14} + \phi_{24} \end{vmatrix}$$

which we next substitute in (2.2). After a somewhat pedestrian calculation one arrives at the form of the Lagrangian expressed in the tensor variables

ϕ_a. It is then found that $\mathscr{L}$ has the form

$$\mathscr{L} = \mathscr{L}^{(1)} + \mathscr{L}^{(0)} \tag{5.1}$$

where $\mathscr{L}^{(1)}$ is a real function of ϕ_μ, $\phi_{\mu\nu}$ (i.e., the field variables describing the spin 1 particle), their first derivatives and their complex conjugates. In a similar way, $\mathscr{L}^{(0)}$ is a real function of ϕ_0, $\phi_{\mu\nu\rho}$ and ϕ_{1234} (the field variables attached to the spin 0 particle), their first derivatives and their complex conjugates. The explicit form of $\mathscr{L}^{(1)}$ is the following

$$\mathscr{L}^{(1)} = -2c[\phi_\mu^*(P_\nu \phi_{\mu\nu} + m_0 c\phi_\mu) - \tfrac{1}{2}\phi_{\mu\nu}^*(m_0 c\phi_{\mu\nu} + P_\mu \phi_\nu - P_\nu \phi_\mu)] + \text{conj.}$$
$$\left(P_\mu \equiv i\hbar\partial_\mu - \frac{q}{c}\mathscr{A}_\mu\right) \tag{5.2}$$

and that of $\mathscr{L}^{(0)}$

$$\begin{aligned}\mathscr{L}^{(0)} = &-2c\phi_{1234}^*(P_1 \phi_{234} - P_2 \phi_{134} + P_3 \phi_{124} - P_4 \phi_{123}) \\ &-2c(\phi_{234}^* P_1 \phi_{1234} - \phi_{134}^* P_2 \phi_{1234} + \phi_{124}^* P_3 \phi_{1234} - \phi_{123}^* P_4 \phi_{1234}) \\ &-2m_0 c^2(-\phi_0^* \phi_0 - \phi_{1234}^* \phi_{1234} + \phi_{123}^* \phi_{123} + \phi_{124}^* \phi_{124} + \phi_{134}^* \phi_{134} \\ &+ \phi_{234}^* \phi_{234}) + \text{conj.}\end{aligned} \tag{5.3}$$

As can be seen, the Lagrange equations arising from the new expression of the Lagrangian,

$$\partial_k \frac{\partial \mathscr{L}}{\partial(\partial_k \phi_a)} + \partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \phi_a)} = \frac{\partial \mathscr{L}}{\partial \phi_a} \qquad (\phi_a = \phi_0, \phi_\mu, \phi_{\mu\nu}, \phi_{\mu\nu\rho}, \phi_{1234}; \mu < \nu < \rho; \; \mu, \nu, \rho = 1, 2, 3, 4)$$

are precisely equations (4.9) and (4.10) which, once introduced in (5.2) and (5.3), yield $\mathscr{L} = \mathscr{L}^{(1)} = \mathscr{L}^{(0)} = 0$, as happened in the non-generalised theory (de Broglie, 1957). From (5.1), (5.2) and (5.3) it results that the study of the two particles with different values of spin (the one with spin 1 and the other with spin 0) can be carried out separately. This means that whenever one wants to study a sole particle of spin 1 it must be assumed that the spin 0 particle does not exist which amounts to taking $\phi_0 = \phi_{1234} = \phi_{\mu\nu\rho} = 0$ and therefore $\mathscr{L}^{(0)} = 0$ and $\mathscr{L} = \mathscr{L}^{(1)}$. Similarly, one must put $\phi_\mu = \phi_{\mu\nu} = 0$ whenever the spin 0 particle alone is considered, hence $\mathscr{L}^{(1)} = 0$ and $\mathscr{L} = \mathscr{L}^{(0)}$. Since the Lagrange equations are linear, one can easily make sure that this separation of the two cases will be retained throughout the theory (in its tensorial form). This is what can be seen from definitions (3.3) and (3.6) given above for the 4-vector current-flow density and the energy-momentum density tensor which, written in tensor variables, take the form

$$s_\mu = \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\mu} \tag{5.4}$$

$$T_{\mu\nu} = \delta_{\mu\nu} \mathscr{L} - \frac{\partial \mathscr{L}}{\partial \mathscr{A}_\mu} \mathscr{A}_\nu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \frac{\partial \phi_a}{\partial x_\nu} - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a^*)} \frac{\partial \phi_a^*}{\partial x_\nu} \tag{5.5}$$

From (5.4) and (5.1) s_μ will then be given by a sum of two vectors, $s_\mu^{(1)}$ and $s_\mu^{(0)}$, the first being a function of the field variables attached to the spin 1 particle and the second containing only the variables describing the spin 0 particle:

$$s_\mu = s_\mu^{(1)} + s_\mu^{(0)}$$

Similar thoughts can occur about $T_{\mu\nu}$ which, in tensor variables, is written

$$T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(0)}$$

More precisely, one obtains from (5.2) and (5.4) the expression of the 4-vector current flow-density of the particle with spin 1,

$$s_\mu^{(1)} = 4q\phi_\nu^* \phi_{\nu\mu} + \text{conj.}$$

and that of the spin 0 particle,

$$s_\mu^{(0)} = 4q\phi_{1234}^* \phi_{\nu\rho\sigma} + \text{conj.} \qquad (\mu\nu\rho\sigma \text{ is a circular permutation of } 1,2,3,4)$$

As for the tensors $T_{\mu\nu}^{(0)}$ and $T_{\mu\nu}^{(1)}$, they are easily obtained from (5.2), (5.3) and (5.5):

$$T_{\mu\nu}^{(1)} = -s_{\mu\nu}^{(1)} \mathscr{A}_\nu - 2ic\hbar(\phi_{\sigma\mu}\, \partial_\nu \phi_\sigma^* + \phi_\sigma\, \partial_\nu \phi_{\sigma\mu}^* - \text{conj.})$$

$$T_{\mu\nu}^{(0)} = -s_\mu^{(0)} \mathscr{A}_\nu - 2ic\hbar\Big(\phi_{\alpha\beta\gamma}^*\, \partial_\nu \phi_{1234} + \phi_{1234}^* \sum_{\substack{\delta,\epsilon,\eta=1\\ \delta<\epsilon<\eta}}^{4} \partial_\nu \phi_{\delta\epsilon\eta} - \text{conj.}\Big)$$

$(\mu\alpha\beta\gamma$ is a circular permutation of $1,2,3,4)$

We shall end the present paper by pointing out that if we introduce in (4.9a) the expression of $\phi_{\mu\nu}$ as yielded by (4.9b), equations (4.9) then assume the equivalent form

$$\left(\partial_\mu + \frac{iq}{\hbar c}\mathscr{A}_\mu\right)\left[\left(\partial_\mu + \frac{iq}{\hbar c}\mathscr{A}_\mu\right)\phi_\nu - \left(\partial_\nu + \frac{iq}{\hbar c}\mathscr{A}_\nu\right)\phi_\mu\right] = \frac{m_0{}^2 c^2}{\hbar^2}\phi_\nu.$$

Now these equations which here describe a special case (that of the spin 1 particle) in the generalised theory of the particle with spin maximum 1, have been introduced in a rather different way by Proca (1936) within the frame of a vector theory, i.e. a theory in which a single 4-vector is employed to describe the particle. Yet the two theories do differ on several points, namely in the Lagrangian and the tensor $T_{\mu\nu}$ (Pereira, 1971). In fact, one must not expect the Lagrangian of the Proca vector theory to be given by (5.2) in which we replace $\phi_{\mu\nu}$ by its expression taken from (4.9b), for $\mathscr{L}^{(1)}$ would then become a function of ϕ_μ, its first and *second* derivatives.

Since the two Lagrangians are different, it follows from (5.5) that the two tensors $T_{\mu\nu}^{(1)}$ are different, too, and as a matter of fact the calculation leads to a symmetric tensor expression for the Proca (1936) tensor.

Now there are some reasons for believing that the physical properties of the spin 1 particle are better explained by an asymmetric tensor (Imbert, 1969). Therefore, it seems that the use of a single 4-vector for the description of the particle may not be justified and must instead be replaced by the introduction of a 4-vector ϕ_μ and a tensor $\phi_{\mu\nu}$.

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